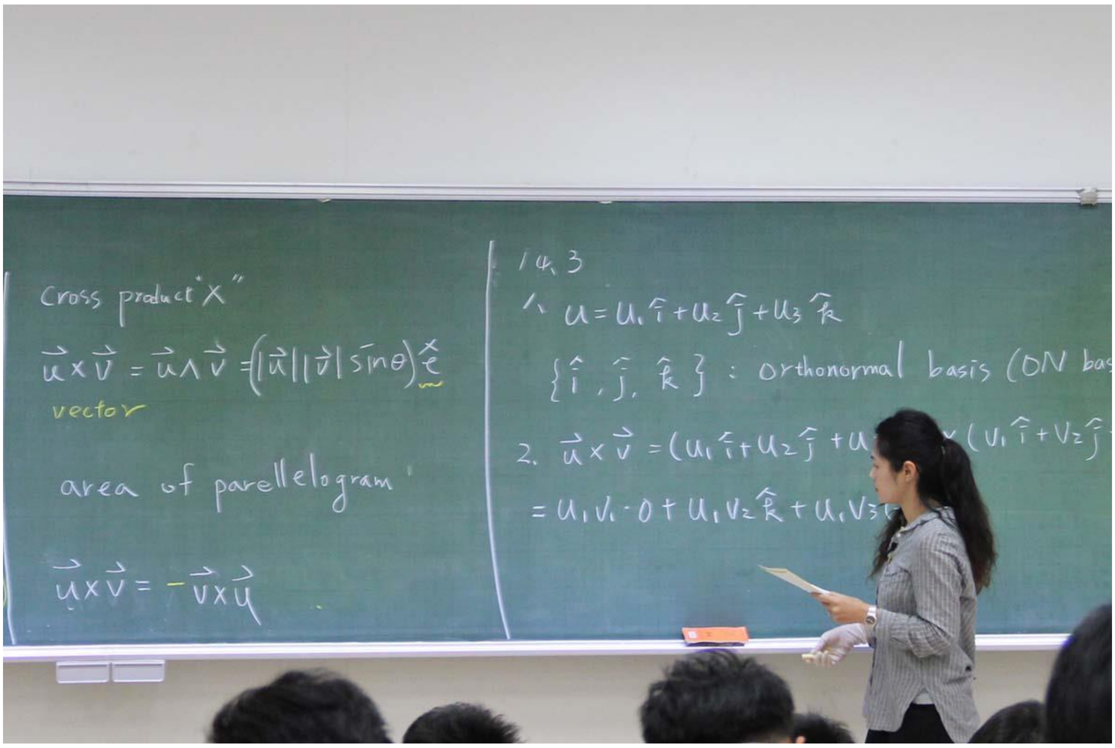
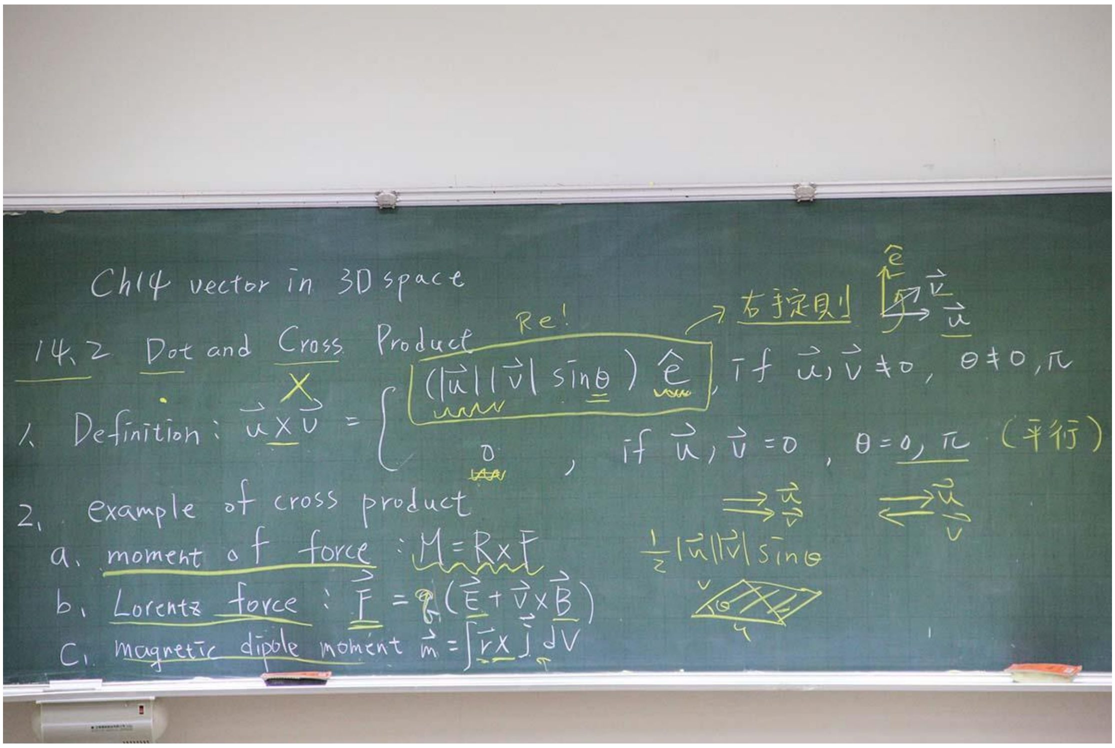
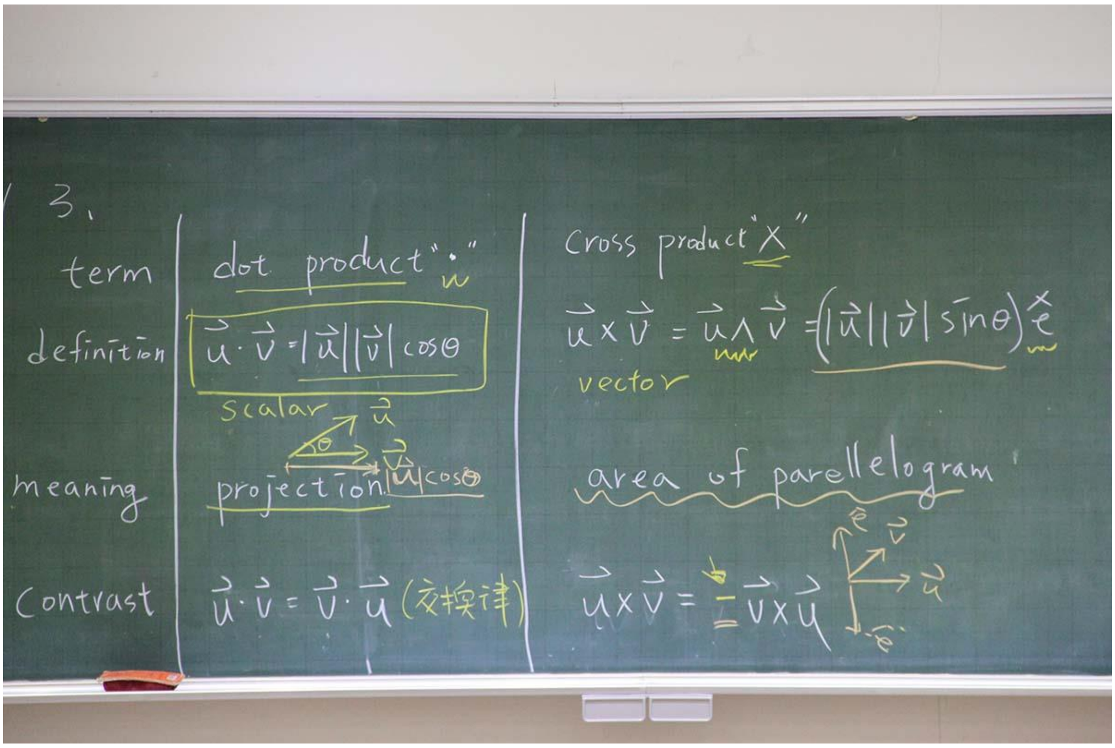
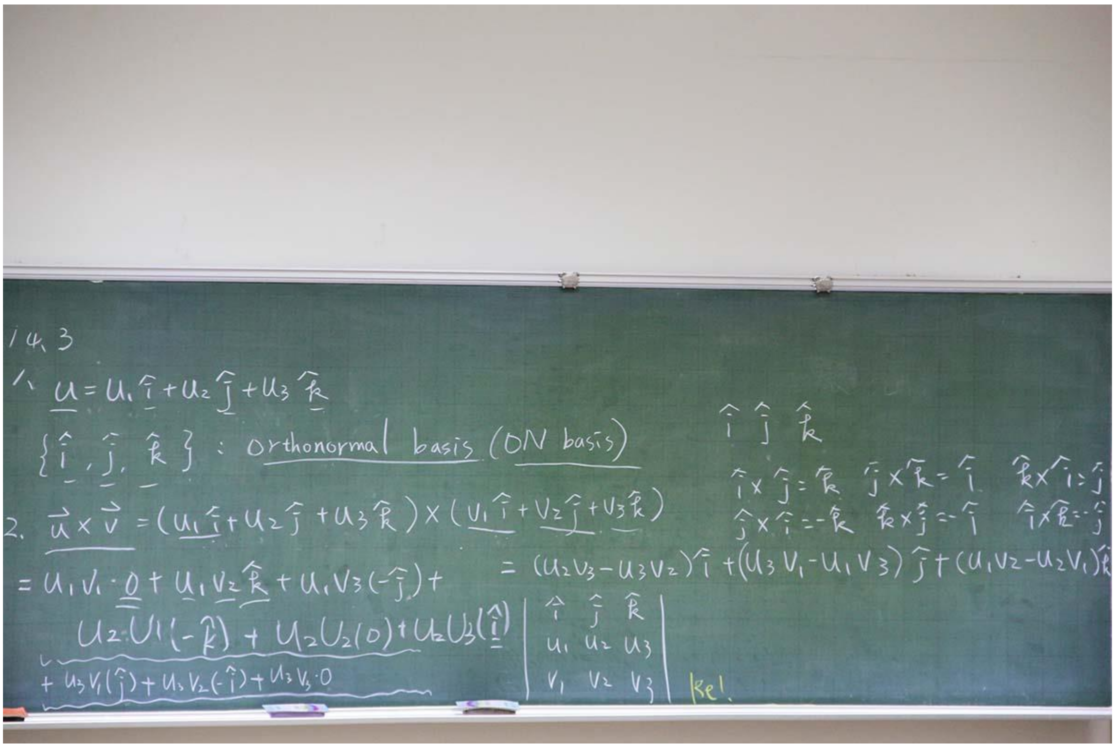


Cross product "×"
$\vec{u}\times\vec{v} = \vec{u}\wedge\vec{v} = (|\vec{u}||\vec{v}|\sin\theta)\hat{e}$
vector
area of parellelogram
$\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$
14.3
1. $u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$
$\{\hat{i}, \hat{j}, \hat{k}\}$: orthonormal basis (ON bas
2. $\vec{u}\times\vec{v} = (u_1\hat{i} + u_2\hat{j} + u$
$= u_1v_1\cdot 0 + u_1v_2\hat{k} + u_1v_3$



Ch14 vector in 3D space
14.2 Dot and Cross Product
Re!
右手定則
1. Definition : $\vec{u}\times\vec{v} = \begin{cases} (|\vec{u}||\vec{v}|\sin\theta)\hat{e}, & \text{if } \vec{u},\vec{v}\neq 0,\ \theta\neq 0,\pi \\ 0, & \text{if } \vec{u},\vec{v}=0,\ \theta=0,\pi \end{cases}$ (平行)
2. example of cross product
a. moment of force : $M = R\times F$
b. Lorentz force : $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$
c. magnetic dipole moment $\vec{m} = \int \vec{r}\times\vec{J}\, dV$
$\frac{1}{2}|\vec{u}||\vec{v}|\sin\theta$


3.
term
dot product "·"
Cross product "X"
definition
u·v = |u||v| cosθ
scalar
u×v = u∧v = (|u||v| sinθ) ê
vector
meaning
projection |u|cosθ
area of parellelogram
Contrast
u·v = v·u (交換律)
u×v = −v×u


14.3
1. u = u1 î + u2 ĵ + u3 k̂
{î, ĵ, k̂} : orthonormal basis (ON basis)
2. u×v = (u1 î + u2 ĵ + u3 k̂) × (v1 î + v2 ĵ + v3 k̂)
= u1v1·0 + u1v2 k̂ + u1v3(−ĵ) +
u2v1(−k̂) + u2v2(0) + u2v3(î)
+ u3v1(ĵ) + u3v2(−î) + u3v3·0
= (u2v3 − u3v2)î + (u3v1 − u1v3)ĵ + (u1v2 − u2v1)k̂
î ĵ k̂
u1 u2 u3
v1 v2 v3
Re!
î ĵ k̂
î×ĵ=k̂ ĵ×k̂=î k̂×î=ĵ
ĵ×î=−k̂ k̂×ĵ=−î î×k̂=−ĵ

14.4 Multiple Products

1. $$u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u\times v\cdot w$$

(1) $|u\cdot(v\times w)|$ = volume of u, v, w parallelepiped

→ proof

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$u\cdot(v\times w) = w\cdot(u\times v) = (u\times v)\cdot w$$

可交換

(2) $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$ Re!

14.5 Differentiation of a vector fxn of a single variable

☆ $R(\tau)$ 和 $R'(\tau)$ 垂直

(1) $\vec{R}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$ in Cartesian coordinate

$R \in$!

$= r(\tau)\,\hat{e}_r + \theta(t)\,\hat{e}_\theta + z(t)\,\hat{k}$ in Cylindrical ;

$= \rho(t)\,\hat{e}_\rho + \phi(t)\,\hat{e}_\phi + \theta(\tau)\,\hat{e}_\theta$ in Spherical ;

2. (1) $\dfrac{d\vec{R}(\tau)}{d\tau} = \vec{R}'(\tau) \equiv \lim\limits_{\Delta\tau \to 0} \dfrac{\vec{R}(\tau+\Delta\tau) - \vec{R}(\tau)}{\Delta\tau}$; (2) $|\vec{R}'(\tau)| = S'(\tau)$ $S(\tau)$: length of arc in ch15 #

tangent vector

<Ex $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k}$

find $r'(t)$; $\hat{e}_t = ?$ $\dfrac{\vec{R}'(\tau)}{|\vec{R}'(\tau)|} = \hat{e}_t$

Sol

$r'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}$

$s'(t) = |r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\hat{e}_t = \dfrac{r'(t)}{s'(t)} = \dfrac{-\sin t}{\sqrt{2}}\,\hat{i} + \dfrac{\cos t}{\sqrt{2}}\,\hat{j} + \dfrac{1}{\sqrt{2}}\,\hat{k}$

訂正講義

14.6 P8

where $\dot{\hat{e}}_r = \cdots = \cdots = \dfrac{\cancel{|\hat{e}_r|}\, d\theta}{\cancel{d\theta}}\hat{e}_\theta$ $\quad \dot{\theta} = \dot{\theta}\,\hat{e}_\theta$

$\hat{e}_r(\theta+d\theta)$

$\rightarrow s = r\cdot\theta = |\dot{\hat{e}}_r|\, d\theta$

$\hat{e}_r(\theta)$

$d\theta$

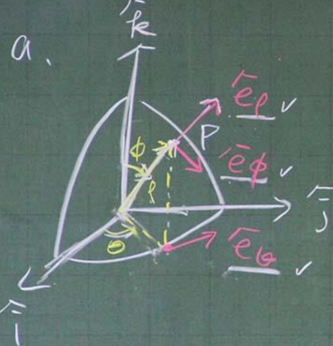
訂正筆記

Cylindrical coordinates

$a(t) = \ddot{r}\hat{e}_r + \dot{r}\hat{e}_\theta\dot{\theta} + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + \ddot{z}\hat{e}_k$

4、Spherical coordinates ($\hat{e}_\rho$ $\hat{e}_\phi$ $\hat{e}_\theta$)

a.



a. ①
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$
Re!

② $\rho = \sqrt{x^2+y^2+z^2}$, $\rho \geq 0$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$, $0 \leq \theta \leq 2\pi$

$\phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $0 \leq \phi \leq \pi$

ρ = const → spherical surface

ϕ = const → conical surface

θ = const ⇒ meridional plane

b. 參考 ch14.6 p711

$\hat{e}_\rho = \hat{e}_\rho(\phi, \theta)$, $\hat{e}_\phi = \hat{e}_\phi(\phi, \theta)$, $\hat{e}_\theta = \hat{e}_\theta(\theta)$

$$\partial \hat{e}_\rho / \partial \rho = \underline{0}, \quad \partial \hat{e}_\rho / \partial \phi = \hat{e}_\phi \ ①, \quad \frac{\partial \hat{e}_\rho}{\partial \theta} = \sin\phi\, \hat{e}_\theta \ ③$$

$$\partial \hat{e}_\phi / \partial \rho = \underline{0}, \quad \partial \hat{e}_\phi / \partial \phi = -\hat{e}_\rho \ ②, \quad \frac{\partial \hat{e}_\phi}{\partial \theta} = \cos\phi\, \hat{e}_\theta \ ④$$

$$\partial \hat{e}_\theta / \partial \rho = \underline{0}, \quad \frac{\partial \hat{e}_\theta}{\partial \phi} = \underline{0}, \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\sin\phi\, \hat{e}_\rho - \cos\phi\, \hat{e}_\phi \ ⑤$$

c. $R(t) = \rho(t)\, \hat{e}_\rho$

$v(t) = \dot{\rho}\, \hat{e}_\rho + \rho \dot{\theta} \sin\theta\, \hat{e}_\theta + \rho \dot{\phi}\, \hat{e}_\phi \rightarrow$ HW

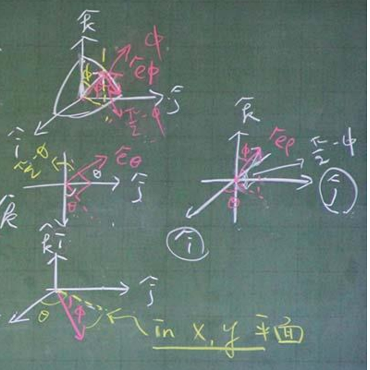
prove by transform method

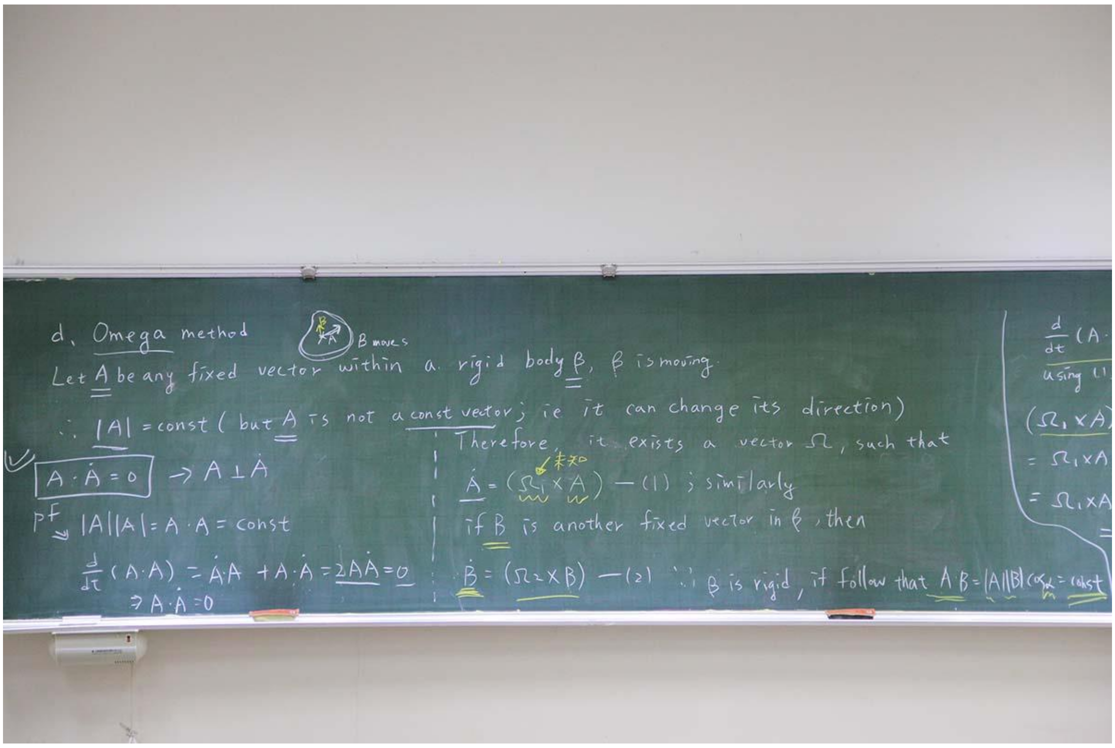
$\hat{e}_\rho = \hat{e}_\rho(\theta, \phi) = \sin\phi\cos\theta\, \hat{i} + \sin\phi\sin\theta\, \hat{j} + \cos\phi\, \hat{k}$

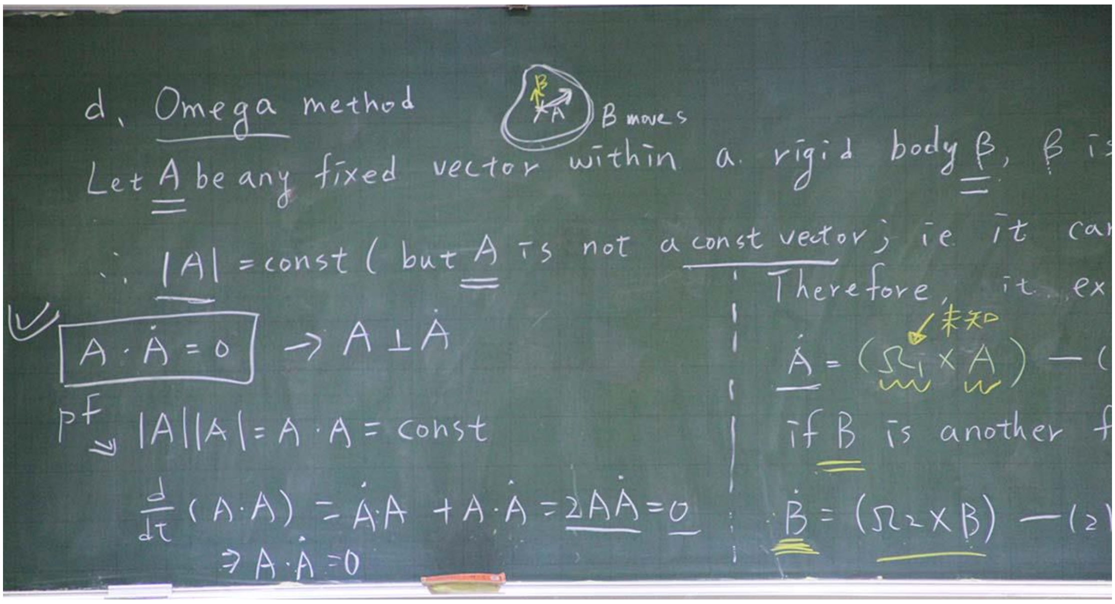
☆ $\hat{e}_\phi = \hat{e}_\phi(\theta, \phi) = \cos\phi\cos\theta\, \hat{i} + \cos\phi\sin\theta\, \hat{j} - \sin\phi\, \hat{k}$

$\hat{e}_\theta = \hat{e}_\theta(\theta) = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}$

prove 1~5 → HW




d、Omega method
B moves
Let A be any fixed vector within a rigid body β, β is moving
∴ |A| = const (but A is not a const vector; ie it can change its direction)
A · Ȧ = 0 → A ⊥ Ȧ
pf |A||A| = A · A = const
d/dt (A·A) = Ȧ·A + A·Ȧ = 2AȦ = 0
⇒ A·Ȧ = 0
Therefore, it exists a vector Ω, such that
未知
Ȧ = (Ω₁ × A) — (1) ; similarly
if B is another fixed vector in β, then
Ḃ = (Ω₂ × B) — (2) ∵ β is rigid, it follow that A·B = |A||B| cos = const
d/dt (A·
using (1
(Ω₁ × A)
= Ω₁ × A
= Ω₁ × A


d、Omega method
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Therefore, it ex
A · Ȧ = 0 → A ⊥ Ȧ
未知
Ȧ = (Ω₁ × A) — (
pf |A||A| = A · A = const
if B is another f
d/dt (A·A) = Ȧ·A + A·Ȧ = 2AȦ = 0
Ḃ = (Ω₂ × B) — (2
⇒ A·Ȧ = 0

... rigid body β, β is moving.

...st vector; ie it can change its direction)

Therefore, it exists a vector Ω, such that

未知

$$\dot{A} = (\Omega_1 \times A) \quad \text{— (1)}$$

; similarly

if B is another fixed vector in β, then

$$\dot{B} = (\Omega_2 \times B) \quad \text{— (2)}$$

$\because$ β is rigid, it follow that $A \cdot B = |A||B| \cos\alpha = \text{const}$

$$\frac{d}{dt}(A \cdot B) = \dot{A} \cdot B + A \cdot \dot{B} = 0 \quad \text{— (3)}$$

using (1) & (2)

$$(\Omega_1 \times A) \cdot B + A \cdot (\Omega_2 \times B)$$

$$= \Omega_1 \times A \cdot B + A \times \Omega_2 \cdot B$$

可互换

$$= \Omega_1 \times A \cdot B - (\Omega_2 \times A) \cdot B$$

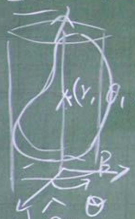
$$= (\Omega_1 - \Omega_2) \times A \cdot B = 0$$

$$\Rightarrow \Omega_1 - \Omega_2 = 0 \Rightarrow \Omega_1 = \Omega_2 = \Omega \ldots$$

<cylindrical coordinates>

$r = r(t)$, $\theta = \theta(t)$, $z = z(t)$

$\Omega = 0 + 0 + \dot{\theta}\hat{e}_z = \dot{\theta}\hat{e}_z$, Let A be $\hat{e}_r$ so eq(4)



θ, z fixed vary r → only translate, $\Omega_r = 0$

r, θ fixed vary z → only translate, $\Omega = 0$

r, z fixed vary θ → anglar velocity $= \dot{\theta}\hat{e}_z$

$$\frac{d\hat{e}_r}{dt} = \Omega \times \hat{e}_r = (\dot{\theta}\hat{e}_z) \times \hat{e}_r = \dot{\theta}\hat{e}_\theta \quad \text{—(5)}$$

$$\frac{d\hat{e}_r(r(t), \theta(t), z(t))}{dt} = \frac{\partial \hat{e}_r}{\partial r}\dot{r} + \frac{\partial \hat{e}_r}{\partial \theta}\dot{\theta} + \frac{\partial \hat{e}_r}{\partial z}\dot{z} \quad \text{—(6)}$$

arbitrary

compare (5) (6)

$$\Rightarrow \frac{\partial \hat{e}_r}{\partial r} = 0, \quad \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial z} = 0$$

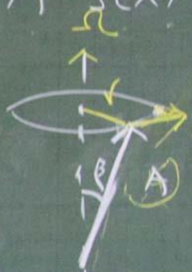
HW 其它 6 个微分

$\dot{A} = \Omega \times A$, $\dot{B} = \Omega \times B$, $\dot{C} = \Omega \times C$ ...

Next, let's under stand what Ω is

$$\dot{A} = \Omega \times A = (|\Omega||A|\sin\beta)\hat{e} = |\Omega|\, r\, \hat{e}$$

ωr → $r\omega$



∴ Ω is the angular velocity vector of β

<spherical coordinate>

$$\Omega = \dot{\phi}\hat{e}_\theta + \dot{\theta}[\cos\phi\, \hat{e}_\rho - \sin\phi\, \hat{e}_\phi]$$

9个微分

HW